Published by Institute of Physics Publishing for SISSA

Received: *September 5, 2005*
Accepted: *September 23, 2005*
Published: *October 11, 2005*

Symmetry of anomalous dimension matrices for colour evolution of hard scattering processes

Michael H. Seymour

School of Physics & Astronomy, University of Manchester
Manchester M13 9PL, U.K., and
Theoretical Physics Group, CERN
CH-1211 Geneva 23, Switzerland
E-mail: mike.seymour@cern.ch

Abstract: In a recent paper, Dokshitzer and Marchesini rederived the anomalous dimension matrix for colour evolution of $gg \to gg$ scattering, first derived by Kidonakis, Oderda and Sterman. They noted a weird symmetry that it possesses under interchange of internal (colour group) and external (scattering angle) degrees of freedom and speculated that this may be related to an embedding into a context that correlates internal and external variables such as string theory.

In this short note, I point out another symmetry possessed by all the colour evolution anomalous dimension matrices calculated to date. It is more prosaic, but equally unexpected, and may also point to the fact that colour evolution might be understood in some deeper theoretical framework. To my knowledge it has not been pointed out elsewhere, or anticipated by any of the authors calculating these matrices. It is simply that, in a suitably chosen colour basis, they are complex symmetric matrices.

Keywords: QCD, Jets.

Following the pioneering work of Botts and Sterman [1], it has long been known that all-orders resummation of virtual corrections to scattering processes can be viewed as an evolution in the colour space of the hard partons. For all but the simplest scattering processes, this colour space is non-trivial and the anomalous dimensions for this evolution are non-commuting matrices, whose dimensionality depends on the colour representations of the hard partons. All the massless $2 \to 2$ anomalous dimension matrices have been calculated in [2]. They range from 2×2 matrices for processes involving only quarks and antiquarks, to 9×9 for $\mathrm{gg} \to \mathrm{gg}$ (although 3 of these dimensions turn out to decouple from physical amplitudes, and 1 is null for $\mathrm{SU}(N_c = 3)$). The anomalous dimension matrices have also been calculated for massive $\mathrm{Q\overline{Q}}$ production processes in [3].

More recently, Dokshitzer and Marchesini have rederived the most complicated case of $\mathrm{gg} \to \mathrm{gg}$ scattering [4] using a physically more transparent method. This allowed them to eliminate the unphysical decoupling dimensions from the start, and to write the final result in a simpler form. In particular, this lays bare a weird symmetry that three of its eigenvalues possess, and they speculated that this symmetry may point to the possibility of a deeper theoretical understanding of colour evolution, perhaps from string theory.

A deeper understanding of the colour evolution is badly needed if we are to extend calculations to higher orders. In general for an n-parton scattering process (i.e. $2 \to (n{-}2)$), there are of order $n!$ colour states and the anomalous dimensions are extremely large matrices. For example, for $\mathrm{gg} \to \mathrm{ggg}$ there are 44 states for general $\mathrm{SU}(N_c)$ and hence we must calculate and diagonalize a 44×44 matrix. Clearly some more fundamental organizing principle is needed to make progress.

In this short note, I would like to point out another symmetry that all the anomalous dimension matrices calculated to date possess. To my knowledge this symmetry was not anticipated by any of the other authors working in the field, and was certainly not by me. It is simply an empirical observation on my part.

One is free to define the anomalous dimension matrix in any convenient colour basis, and different calculations in the literature have used different bases. However, it is convenient to use an orthogonal basis in which the lowest order soft matrix (in the nomenclature of refs. [2, 3], or the metric tensor of the colour space in the nomenclature of ref. [4]) is diagonal, and all published results have done this.

As a concrete example, I use the result for $\mathrm{gg} \to \mathrm{gg}$ using the nomenclature of Kidonakis, Sterman et al, and the simplified form of the result from Dokshitzer and Marchesini. They use a set of s-channel projectors as their colour basis, and therefore have a lowest order colour matrix of

$$\mathrm{S} = \begin{pmatrix} K_{8_a} & & & & & \\ & K_{10} & & & & \\ & & K_1 & & & \\ & & & K_{8_s} & & \\ & & & & K_{27} & \\ & & & & & K_0 \end{pmatrix}, \tag{1}$$

JHEP10(2005)029

where K_α are the dimensionalities of the corresponding representations,

$$\begin{aligned} K_1 &= 1\,, \qquad K_{8_a} = K_{8_s} = N_c^2 - 1\,, \qquad & K_{10} &= 2 \cdot \frac{(N_c^2-1)(N_c^2-4)}{4}\,, \\ K_{27} &= \frac{N_c^2(N_c-1)(N_c+3)}{4}\,, & K_0 &= \frac{N_c^2(N_c+1)(N_c-3)}{4}\,, \end{aligned} \tag{2}$$

where the labels are the SU(3) values (and strictly speaking 10 should be $10 + \overline{10}$). They find an anomalous dimension matrix of

$$\Gamma = \begin{pmatrix} \frac{3}{2} & 0 & -2b & -\frac{1}{2}b & -\frac{2}{N_c^2}b & -\frac{2}{N_c^2}b \\ 0 & 1 & 0 & -b & -\frac{(N_c+1)(N_c-2)}{N_c^2}b & -\frac{(N_c-1)(N_c+2)}{N_c^2}b \\ -\frac{2}{N_c^2-1}b & 0 & 2 & 0 & 0 & 0 \\ -\frac{1}{2}b & -\frac{2}{N_c^2-4}b & 0 & \frac{3}{2} & 0 & 0 \\ -\frac{N_c+3}{2(N_c+1)}b & -\frac{N_c+3}{2(N_c+2)}b & 0 & 0 & \frac{N_c-1}{N_c} & 0 \\ -\frac{N_c-3}{2(N_c-1)}b & -\frac{N_c-3}{2(N_c-2)}b & 0 & 0 & 0 & \frac{N_c+1}{N_c} \end{pmatrix}, \tag{3}$$

where

$$b = \frac{T-U}{T+U}\,, \qquad T = \ln\frac{s}{|t|} - i\pi\,, \qquad U = \ln\frac{s}{|u|} - i\pi\,. \tag{4}$$

The symmetry I wish to point out is rather simple. It is that in an orthonormal basis, i.e. one in which the lowest order soft matrix is the identity matrix, the anomalous dimension matrix is symmetric. Returning to the concrete example of gg $\to$ gg, transforming to an orthonormal basis, i.e. normalizing the basis vectors, effectively multiplies the $\alpha\beta$ element of Γ by $\sqrt{K_\beta/K_\alpha}$, yielding

$$\Gamma = \begin{pmatrix} \frac{3}{2} & 0 & -\frac{2}{\sqrt{N_c^2-1}}b & -\frac{1}{2}b & -\frac{1}{N_c}\sqrt{\frac{N_c+3}{N_c+1}}b & -\frac{1}{N_c}\sqrt{\frac{N_c-3}{N_c-1}}b \\ 0 & 1 & 0 & -\sqrt{\frac{2}{N_c^2-4}}b & -\sqrt{\frac{(N_c+3)(N_c+1)(N_c-2)}{2N_c^2(N_c+2)}}b & -\sqrt{\frac{(N_c-3)(N_c-1)(N_c+2)}{2N_c^2(N_c-2)}}b \\ -\frac{2}{\sqrt{N_c^2-1}}b & 0 & 2 & 0 & 0 & 0 \\ -\frac{1}{2}b & -\sqrt{\frac{2}{N_c^2-4}}b & 0 & \frac{3}{2} & 0 & 0 \\ -\frac{1}{N_c}\sqrt{\frac{N_c+3}{N_c+1}}b & -\sqrt{\frac{(N_c+3)(N_c+1)(N_c-2)}{2N_c^2(N_c+2)}}b & 0 & 0 & \frac{N_c-1}{N_c} & 0 \\ -\frac{1}{N_c}\sqrt{\frac{N_c-3}{N_c-1}}b & -\sqrt{\frac{(N_c-3)(N_c-1)(N_c+2)}{2N_c^2(N_c-2)}}b & 0 & 0 & 0 & \frac{N_c+1}{N_c} \end{pmatrix}, \tag{5}$$

a manifestly symmetric matrix.

This is just one example, but I have found it to be true of all the anomalous dimension matrices calculated to date. It is true not only for the anomalous dimension matrices of the virtual matrix elements, but also those for energy flow observables, whether defined for a slice in rapidity [5], a patch in rapidity and azimuth [6], or for a region of the event defined by the $k_\perp$ algorithm [7]. In a forthcoming paper [8], we have calculated an anomalous dimension matrix for the colour evolution of a five-parton system, necessary to describe a $2 \to 3$ process, as far as I know for the first time. It also has this symmetric structure in an orthonormal basis.

It is worth emphasizing that the anomalous dimension matrix is complex, so the fact that it is symmetric is not equivalent to it being hermitean, a property that might have

JHEP10(2005)029

been slightly less surprising. If one wished to view the anomalous dimension matrix as a transition amplitude for an effective theory in colour space, one might imagine that time reversal symmetry of this theory would lead to a hermitean matrix, but the fact that it is symmetric rather than hermitean does not allow this interpretation.

I would like to close this short note by recalling that I have no explanation for why the anomalous dimension matrices of colour evolution should be symmetric. This surprising result may turn out to be trivial, or may point the way to a deeper understanding of colour evolution.

References

[1] J. Botts and G. Sterman, *Hard elastic scattering in QCD: leading behavior*, *Nucl. Phys.* **B 325** (1989) 62.

[2] N. Kidonakis, G. Oderda and G. Sterman, *Evolution of color exchange in QCD hard scattering*, *Nucl. Phys.* **B 531** (1998) 365 [hep-ph/9803241].

[3] N. Kidonakis and G. Sterman, *Resummation for QCD hard scattering*, *Nucl. Phys.* **B 505** (1997) 321 [hep-ph/9705234].

[4] Yu.L. Dokshitzer and G. Marchesini, *Hadron collisions and the fifth form factor*, hep-ph/0508130.

[5] G. Oderda, *Dijet rapidity gaps in photoproduction from perturbative QCD*, *Phys. Rev.* **D 61** (2000) 014004 [hep-ph/9903240].

[6] C.F. Berger, T. Kúcs and G. Sterman, *Energy flow in interjet radiation*, *Phys. Rev.* **D 65** (2002) 094031 [hep-ph/0110004].

[7] R.B. Appleby and M.H. Seymour, *The resummation of inter-jet energy flow for gaps-between-jets processes at HERA*, *JHEP* **09** (2003) 056 [hep-ph/0308086].

[8] J.R. Forshaw, A. Kyrieleis and M.H. Seymour, *On the rôle of Coulomb phase terms in gaps between jets cross sections*, in preparation.